

Abelian quotients of subgroups of the mapping class group and higher Prym representations

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Abstract

A well-known conjecture asserts that the mapping class group of a surface (possibly with punctures/boundary) does not virtually surject onto \mathbb{Z} if the genus of the surface is large. We prove that if this conjecture holds for some genus, then it also holds for all larger genera. We also prove that if there is a counterexample to this conjecture, then there must be a counterexample of a particularly simple form. We prove these results by relating the conjecture to a family of linear representations of the mapping class group that we call the higher Prym representations. They generalize the classical symplectic representation.

1 Introduction

Let $\Sigma_{g,n}^p$ be an orientable genus g surface with n boundary components and p punctures and let $\text{Mod}_{g,n}^p$ be the *mapping class group* of $\Sigma_{g,n}^p$. This is the group of orientation-preserving homeomorphisms of $\Sigma_{g,n}^p$ that fix the boundary and punctures pointwise modulo isotopies that fix the boundary and punctures pointwise. The group $\text{Mod}_{g,n}^p$ plays an important role in areas ranging from low-dimensional topology to algebraic geometry. See [10] for a survey.

There is a fruitful analogy between the mapping class group and lattices in higher rank Lie groups. Such lattices satisfy Kazhdan's property (T), and as a consequence they do not virtually surject onto \mathbb{Z} . In other words, if Γ is a finite-index subgroup of a lattice in a higher-rank Lie group, then $H_1(\Gamma; \mathbb{Q}) = 0$ (see, e.g., [24, Theorems 7.1.4 and 7.1.7]). The starting point for this paper is the following well-known conjecture of Ivanov (see, e.g., [17, Problem 2.11.A] and [15, §7]). It asserts that something similar happens for $\text{Mod}_{g,n}^p$. See the end of this introduction for a summary of the previous literature on it.

Conjecture 1.1. *For some $g \geq 3$ and $n, p \geq 0$, let $\Gamma < \text{Mod}_{g,n}^p$ be a finite-index subgroup. Then $H_1(\Gamma; \mathbb{Q}) = 0$.*

Remark. J. Anderson [2] recently proved that $\text{Mod}_{g,n}^p$ does *not* satisfy Kazhdan's property (T) for $g \geq 2$.

Remark. McCarthy [19] and Taherkhani [23] proved that Conjecture 1.1 fails for $g = 2$.

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Stability theorem. We have two main theorems. The first will interest those who believe that Conjecture 1.1 is true and the second will interest those who do not. Let $\text{NVSZ}(g)$ stand for the assertion that Conjecture 1.1 holds for $\Sigma_{g,n}^p$ for all $n, p \geq 0$. Our first result is as follows.

Theorem A (Stability). *Assume that $\text{NVSZ}(G)$ holds for some $G \geq 3$. Then $\text{NVSZ}(g)$ holds for all $g \geq G$.*

In particular, Theorem A says that to prove Conjecture 1.1, it is enough to deal with the genus 3 case.

Easy counterexamples. Our second main theorem says that if Conjecture 1.1 fails, then there must be a counterexample of a particularly simple form. If $g \geq 2$ and $p \geq 1$, then there is a well-known short exact sequence called the *Birman exact sequence* (see §2.2.2) that takes the form

$$1 \longrightarrow \pi_1(\Sigma_{g,n}^{p-1}) \longrightarrow \text{Mod}_{g,n}^p \longrightarrow \text{Mod}_{g,n}^{p-1} \longrightarrow 1.$$

The terms in this exact sequence are as follows.

- The map $\text{Mod}_{g,n}^p \rightarrow \text{Mod}_{g,n}^{p-1}$ comes from filling in a puncture x on $\Sigma_{g,n}^p$.
- $\pi_1(\Sigma_{g,n}^{p-1})$ is embedded in $\text{Mod}_{g,n}^p$ as mapping classes the “push” the puncture x around the surface $\Sigma_{g,n}^{p-1}$. It is often known as the “point-pushing” subgroup of $\text{Mod}_{g,n}^p$.

If $n \geq 1$, then this exact sequence splits, so we have a semidirect product decomposition

$$\text{Mod}_{g,n}^p \cong \pi_1(\Sigma_{g,n}^{p-1}) \rtimes \text{Mod}_{g,n}^{p-1}.$$

If $K < \pi_1(\Sigma_{g,n}^{p-1})$ and $G < \text{Mod}_{g,n}^{p-1}$ are finite-index subgroups such that G normalizes K , then we obtain a finite-index subgroup

$$\Gamma = K \rtimes G < \text{Mod}_{g,n}^p.$$

Observe now that

$$H_1(\Gamma; \mathbb{Q}) \cong H_1(G; \mathbb{Q}) \oplus (H_1(K; \mathbb{Q}))_G,$$

where $(H_1(K; \mathbb{Q}))_G$ are the *coinvariants* of the action of G on $H_1(K; \mathbb{Q})$, i.e. the quotient of $H_1(K; \mathbb{Q})$ by the subspace generated by the set $\{x - g(x) \mid x \in H_1(K; \mathbb{Q}), g \in G\}$. In particular, if $(H_1(K; \mathbb{Q}))_G \neq 0$, then $H_1(\Gamma; \mathbb{Q}) \neq 0$, in which case we will say that Γ surjects onto \mathbb{Z} by the *finite-index point-pushing construction*.

Theorem B (Easy counterexamples). *If $\text{NVSZ}(g)$ fails, then there is a counterexample to $\text{NVSZ}(g-1)$ that surjects onto \mathbb{Z} by the finite-index point-pushing construction.*

Higher Prym representations. We prove Theorems A and B by relating Conjecture 1.1 to a family of linear representations of $\text{Mod}_{g,n}^p$ that we call the higher Prym representations. As motivation, observe that $\text{Mod}_{g,n}^p$ acts on $H_1(\Sigma_{g,n}^p; \mathbb{Z})$ and preserves the algebraic intersection form. If $n + p \leq 1$, then this is a nondegenerate alternating form, so in these cases we get a representation $\text{Mod}_{g,n}^p \rightarrow \text{Sp}_{2g}(\mathbb{Z})$. This representation plays a fundamental role in the study of $\text{Mod}_{g,n}^p$.

We generalize the symplectic representation of the mapping class group in the following way. Recall that a subgroup G' of a group G is *characteristic* if $f(G') \subset G'$ for all $f \in \text{Aut}(G)$. Fix a basepoint $v_0 \in \Sigma_{g,n}^p$. Regarding the basepoint v_0 as a puncture, the group $\text{Mod}_{g,n}^{p+1}$ acts on $\pi_1(\Sigma_{g,n}^p, v_0)$. If $K < \pi_1(\Sigma_{g,n}^p, v_0)$ is a finite-index characteristic subgroup, then $\text{Mod}_{g,n}^{p+1}$ acts on the finite-dimensional vector space $H_1(K; \mathbb{Q})$.

If S is the finite cover of $\Sigma_{g,n}^p$ corresponding to K , then $H_1(K; \mathbb{Q}) \cong H_1(S; \mathbb{Q})$. The action of $\text{Mod}_{g,n}^{p+1}$ on $H_1(K; \mathbb{Q})$ arises from lifting mapping classes on $\Sigma_{g,n}^{p+1}$ to the cover S (observe that this uses the fact that $\text{Mod}_{g,n}^{p+1}$ fixes the basepoint $v_0 \in \Sigma_{g,n}^p$ – if there were no basepoint, then one could only lift mapping classes modulo the action of the deck group). The *boundary subspace* B of $H_1(K; \mathbb{Q})$ is the subspace spanned by the homology classes of the boundary components of S and loops freely homotopic into the punctures of S . Define $V_K = H_1(K; \mathbb{Q})/B$. It is clear that $\text{Mod}_{g,n}^{p+1}$ preserves B , so we obtain an action of $\text{Mod}_{g,n}^{p+1}$ on V_K . We will call the resulting linear representation $\text{Mod}_{g,n}^{p+1} \rightarrow \text{Aut}(V_K)$ a *higher Prym representation*.

Remark. For $K < \pi_1(\Sigma_g, v_0)$ such that $\pi_1(\Sigma_g, v_0)/K$ is abelian, the action of Mod_g^1 on $H_1(K; \mathbb{Q})$ was studied by Looijenga in [18] (though he arranged the technical details a little differently). He called his representations Prym representations because they are related to Prym varieties.

Remark. The vector space V_K is the first homology group of the closed surface that results from gluing discs to all the boundary components of S and filling in all the punctures of S . This implies that V_K has a nondegenerate intersection pairing that is preserved by $\text{Mod}_{g,n}^{p+1}$, so the image of a higher Prym representation lies in the symplectic group.

Remark. Representations of $\text{Aut}(F_n)$ similar to the higher Prym representations were studied by Grunewald and Lubotzky [11].

Nontriviality of the Prym representations. We make the following conjecture about the higher Prym representations.

Conjecture 1.2. *Fix $g \geq 2$ and $n, p \geq 0$. Let $K < \pi_1(\Sigma_{g,n}^p)$ be a finite-index characteristic subgroup. Then for all nonzero vectors $v \in V_K$, the $\text{Mod}_{g,n}^{p+1}$ -orbit of v is infinite.*

Equivalently, none of the higher Prym representations have subrepresentations that factor through a finite group.

Remark. When $\pi_1(\Sigma_{g,n}^p)/K$ is abelian, Conjecture 1.2 follows easily from Looijenga's results in [18].

Remark. Conjecture 1.2 is false for $g = 0$ and $g = 1$. The case $g = 0$ appears (in a different language) in work of McMullen [20]. Translated into our language, [20, Theorem

8.1] gives a list of finite-index subgroups $K < \pi_1(\Sigma_{0,0}^p)$ such that the subgroups $\Gamma_K < \text{Mod}_{0,0}^{p+1}$ preserving K have nontrivial finite orbits in V_K . Taking the intersection of all the $\text{Mod}_{0,0}^{p+1}$ -translates of one of these K gives a characteristic subgroup of $\pi_1(\Sigma_{0,0}^p)$ which is a counterexample to Conjecture 1.2. For the case $g = 1$, it is not hard to construct a finite-index normal subgroup $K < \pi_1(\Sigma_{1,0}^1)$ whose stabilizer $\Gamma_K < \text{Mod}_{1,0}^2$ has a nontrivial finite orbit in V_K . The quotient $\pi_1(\Sigma_{1,0}^1)/K$ is the 8-element quaternion group. Again, taking the intersection of all the $\text{Mod}_{1,0}^2$ -translates of K gives a counterexample to Conjecture 1.2.

Relation between conjectures. Conjectures 1.1 and 1.2 appear quite different. However, it turns out that they are essentially equivalent. Let $\text{NVSZ}(g, n, p)$ stand for the assertion that Conjecture 1.1 is true for $\Sigma_{g,n}^p$ and let $\text{NFO}(g, n, p)$ stand for the assertion that Conjecture 1.2 is true for $\Sigma_{g,n}^p$.

Theorem C. *Fix $g \geq 3$ and $p \geq 0$.*

- $\text{NFO}(g-1, n+1, p)$ *implies* $\text{NVSZ}(g, n, p)$ *for* $n \geq 0$.
- $\text{NVSZ}(g, n, p+1)$ *implies* $\text{NFO}(g, n, p)$ *for* $n \geq 1$.

Remark. It is easy to see that $\text{NFO}(g, n, p)$ implies $\text{NFO}(g, n', p')$ for $n' \leq n$ and $p' \leq p$, so Theorem C also shows that $\text{NVSZ}(g, 1, p)$ implies $\text{NFO}(g, 0, p)$ for $g \geq 3$ and $p \geq 0$.

Derivation of Theorems A and B from Theorem C. Theorem C immediately implies Theorems A and B. First, we can apply Theorem C twice and see that if $g \geq 3$ and $n, p \geq 0$, then

$$\text{NVSZ}(g, n+1, p+1) \implies \text{NFO}(g, n+1, p) \implies \text{NVSZ}(g+1, n, p).$$

Iterating this, we see that if $\text{NVSZ}(G, n + (g - G), p + (g - G))$ is true for some $G \geq 3$ and $g \geq G$ and $n, p \geq 0$, then $\text{NVSZ}(g, n, p)$ is true. In other words, $\text{NVSZ}(G)$ implies $\text{NVSZ}(g)$ for $g \geq G$.

As far as Theorem B goes, we can again apply Theorem C twice and see that if $g \geq 3$ and $n, p \geq 0$, then

$$\neg \text{NVSZ}(g, n, p) \implies \neg \text{NFO}(g-1, n+1, p) \implies \neg \text{NVSZ}(g-1, n+1, p+1).$$

The proof of the second implication here actually produces a counterexample to $\text{NVSZ}(g-1, n+1, p+1)$ using the finite-index point-pushing construction. The input for this construction is obtained from a counterexample to the assertion $\text{NFO}(g-1, n+1, p)$. See §4 for more details.

Remark. An amusing property of the above derivation is that it works whether Conjecture 1.2 holds or not.

History. Fix some $g \geq 3$. There are only a few known examples of finite-index subgroups $\Gamma < \text{Mod}_{g,n}^p$ for which $H_1(\Gamma; \mathbb{Q})$ is known to vanish. In [12], Hain verifies this for Γ that contain the *Torelli group*, which is the kernel of the action of $\text{Mod}_{g,n}^p$ on $H_1(\Sigma_{g,0}^0; \mathbb{Z})$. This was later generalized to some deeper subgroups by Boggi [6] and the first author [22].

A related result, which was proven independently by the first author [22] and Bridson [7], says that if $\Gamma < \text{Mod}_{g,n}^p$ is a finite index subgroup and $T_\gamma \in \text{Mod}_{g,n}^p$ is a Dehn twist about a simple closed curve γ , then the image of T_γ^k in $H_1(\Gamma; \mathbb{Q})$ vanishes for any k such that $T_\gamma^k \in \Gamma$. This theorem will play an important role in our proof of Theorem C.

We finally should mention Boggi's recent work on the congruence subgroup problem for $\text{Mod}_{g,n}^p$ (see [5]). The congruence subgroup problem gives a conjectural classification of all finite-index subgroups of $\text{Mod}_{g,n}^p$. Though Boggi's proof of the congruence subgroup problem itself appears to be fatally flawed, he does give a correct proof of the following beautiful result. Let $\mathcal{C}_{g,n}^p$ be the curve complex for $\Sigma_{g,n}^p$ (see §2.2.3 for details). A theorem of Harer says that $H_k(\mathcal{C}_{g,n}^p; \mathbb{Z}) = 0$ for k in some range. Boggi proved that we also have $H_k(\mathcal{C}_{g,n}^p/\Gamma; \mathbb{Q}) = 0$ for k in this same range for Γ a finite-index subgroup of $\text{Mod}_{g,n}^p$. This result, which was proven using the theory of weights on the cohomology of algebraic varieties [9], plays a fundamental role in our proof of Theorem C.

Outline. We begin in §2 with a discussion of some preliminary material about mapping class groups and group homology. Next, in §3 we prove the portion of Theorem C asserting that Conjecture 1.2 implies Conjecture 1.1 (see Theorem 3.3). Finally, in §4 we prove the portion of Theorem C asserting that Conjecture 1.1 implies Conjecture 1.2.

2 Preliminaries

This section has two parts. Some facts about group homology are reviewed in §2.1 and some background about the mapping class group is discussed in §2.2.

2.1 Group homology

A good reference for this material is [8]. We begin with some notation.

Definition. Let G be a group and let M be a vector space upon which G acts.

- The *invariants* of G acting on M are $M^G = \{x \in M \mid g(x) = x \text{ for all } g \in G\}$.
- The *coinvariants* of G acting on M are $M_G = M/K$, where $K < M$ is the subspace spanned by the set $\{x - g(x) \mid x \in M, g \in G\}$.

The invariants and coinvariants are related by the following lemma, whose proof is an easy exercise.

Lemma 2.1. *Let G be a group and M be a G -vector space. Let M^* denote the dual of M . Then $(M_G)^* \cong (M^*)^G$.*

The next lemma we need is the following, which is a standard consequence of the existence of the *transfer map* (see, e.g., [8, Chapter III.9]).

Lemma 2.2. *If G_1 is a finite index subgroup of G_2 , then the map $H_1(G_1; \mathbb{Q}) \rightarrow H_1(G_2; \mathbb{Q})$ is a surjection.*

If G_1 is a normal subgroup of G_2 , then G_2 acts on G_1 and we have the following more precise lemma, which is an immediate consequence of the Hochschild-Serre spectral sequence.

Lemma 2.3. *If G_1 is a finite index normal subgroup of G_2 , then we have $H_1(G_2; \mathbb{Q}) \cong (H_1(G_1; \mathbb{Q}))_{G_2}$.*

The final result we need gives a decomposition of $H_1(G)$ for G acting on a space X . We need the following definition.

Definition. A group G acts on a simplicial complex X *without rotations* if for all simplices s of X , the stabilizer subgroup G_s stabilizes s pointwise.

The theorem we need is as follows. It follows easily from the two spectral sequences given in [8, Chapter VII.7].

Theorem 2.4. *Let a group G act on a connected simplicial complex X without rotations. Fix a ring R . Assume that $H_1(X; R) = H_1(X/G; R) = 0$. Then the natural map*

$$\bigoplus_{v \in X^{(0)}} H_1(G_v; R) \longrightarrow H_1(G; R)$$

is a surjection.

2.2 Surfaces and mapping class groups

A good reference for this material is [10], which contains proofs of all statements for which we do not give proofs.

2.2.1 Embedding one surface into another

Assume that $g \geq 2$ and $n, p \geq 0$. Let S be a subsurface of $\Sigma_{g,n}^p$, and denote by $\text{Mod}(S)$ the mapping class group of S . There is then an induced map $\phi : \text{Mod}(S) \rightarrow \text{Mod}_{g,n}^p$ obtained by extending mapping classes by the identity over $\Sigma_{g,n}^p \setminus S$. This map is usually injective. However, in the following situations it is not injective.

- If $\Sigma_{g,n}^p \setminus \text{Int}(S)$ is a punctured disc with boundary component β , then we have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{Mod}(S) \longrightarrow \text{Mod}_{g,n}^p \longrightarrow 1. \quad (1)$$

Here the kernel \mathbb{Z} is generated by the Dehn twist T_β .

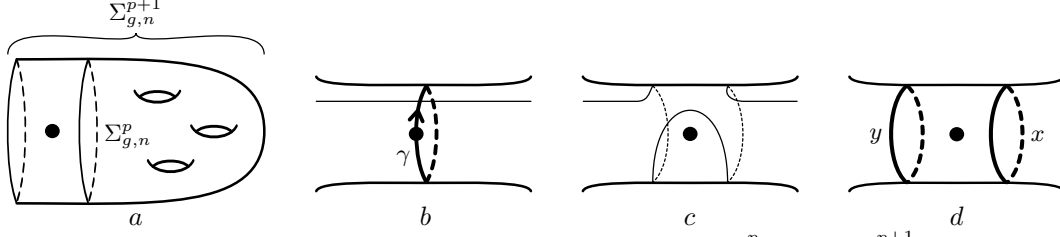


Figure 1: a. We can split the Birman exact sequence by a map $\text{Mod}_{g,n}^p \hookrightarrow \text{Mod}_{g,n}^{p+1}$ as shown. b. A simple closed curve $\gamma \in \pi_1(\Sigma_{g,n}^p)$. c. The result of pushing the basepoint around γ . d. The associated element of $\text{Mod}_{g,n}^{p+1}$ is $T_x T_y^{-1}$, where x and y are as shown.

- If $\Sigma_{g,n}^p \setminus \text{Int}(S)$ is an annulus with boundary components x and y and both x and y lie in S , then we have a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \text{Mod}(S) \longrightarrow (\text{Mod}_{g,n}^p)_\gamma \longrightarrow 1. \quad (2)$$

Here γ is a simple closed curve forming the core of the annulus $\Sigma_{g,n}^p \setminus \text{Int}(S)$ and $(\text{Mod}_{g,n}^p)_\gamma$ is the subgroup of $\text{Mod}_{g,n}^p$ consisting of mapping classes that fix the isotopy class of γ (as an oriented curve – elements of $(\text{Mod}_{g,n}^p)_\gamma$ cannot reverse the orientation of γ). Also, \mathbb{Z} is generated by $T_x T_y^{-1}$.

2.2.2 Surfaces with boundary and the Birman exact sequence

Assume that $g \geq 2$ and $n, p \geq 0$. As was discussed in §1, we have the Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g,n}^p) \longrightarrow \text{Mod}_{g,n}^{p+1} \longrightarrow \text{Mod}_{g,n}^p \longrightarrow 1,$$

where the map $\text{Mod}_{g,n}^{p+1} \longrightarrow \text{Mod}_{g,n}^p$ comes from filling in a puncture x of $\Sigma_{g,n}^{p+1}$ and $\pi_1(\Sigma_{g,n}^p)$ is embedded in $\text{Mod}_{g,n}^{p+1}$ as mapping classes that “push” x around loops in $\Sigma_{g,n}^p$. If $n \geq 1$, then this exact sequence splits via a map $\text{Mod}_{g,n}^p \hookrightarrow \text{Mod}_{g,n}^{p+1}$ induced by an embedding $\Sigma_{g,n}^p \hookrightarrow \Sigma_{g,n}^{p+1}$ such that $\Sigma_{g,n}^{p+1} \setminus \text{Int} \Sigma_{g,n}^p$ is homeomorphic to $\Sigma_{0,2}^1$. This originally appeared in [4], and a suitable textbook reference is [10].

We will need the following standard lemma about the mapping classes associated to certain elements of $\pi_1(\Sigma_{g,n}^p)$.

Lemma 2.5. *Consider $\gamma \in \pi_1(\Sigma_{g,n}^p)$ that can be realized by a simple closed curve. Then the element of $\text{Mod}_{g,n}^{p+1}$ associated to γ is a multitwist.*

For example, if $\gamma \in \pi_1(\Sigma_{g,n}^p)$ is as indicated in Figure 1.b, then as indicated in Figure 1.c the mapping class associated to γ equals $T_x T_y^{-1}$, where x and y are the curves indicated in Figure 1.d.

To help us recognize elements of $\pi_1(\Sigma_{g,n}^p)$ that can be realized by simple closed curves, the following well-known lemma will be useful.

Lemma 2.6. *Fix $g, n, p \geq 0$, and let $v_0 \in \text{Int}(\Sigma_{g,n}^p)$. Consider $\gamma \in \pi_1(\Sigma_{g,n}^p, v_0)$ such that γ is freely homotopic to a simple closed curve. Then γ can be realized by a based simple closed curve.*

Proof. The conditions imply that there is some $\gamma' \in \pi_1(\Sigma_{g,n}^p, v_0)$ that can be realized by a simple closed curve such that γ' is conjugate to γ . Puncturing $\Sigma_{g,n}^p$ at v_0 , the group $\text{Mod}_{g,n}^{p+1}$ acts on $\pi_1(\Sigma_{g,n}^p, v_0)$. The Dehn-Nielsen-Baer theorem (see, e.g., [10]) implies that the image of the resulting map $\text{Mod}_{g,n}^{p+1} \rightarrow \text{Aut}(\pi_1(\Sigma_{g,n}^p, v_0))$ contains all inner automorphisms. This implies that there is some $f \in \text{Mod}_{g,n}^{p+1}$ such that $f(\gamma') = \gamma$. Since γ' can be realized by a simple closed curve, we conclude that γ can as well. \square

2.2.3 The curve complex

We will need some results about the curve complex. This space, which was introduced by Harvey [14], plays a role in the study of the mapping class group analogous to the role of the Tits building of an algebraic group. We start with the definition.

Definition. Fix $g, n, p \geq 0$. A simple closed curve on $\Sigma_{g,n}^p$ is *nontrivial* if it is not homotopic to a point, a puncture, or a boundary component. The *curve complex* on $\Sigma_{g,n}^p$, denoted $\mathcal{C}_{g,n}^p$, is the simplicial complex whose k -simplices are sets $\{\gamma_0, \dots, \gamma_k\}$ of homotopy classes of nontrivial simple closed curves on $\Sigma_{g,n}^p$ that can be realized disjointly.

We will also need the following space.

Definition. The *nonseparating curve complex*, denoted $\mathcal{CNS}_{g,n}^p$, is the subcomplex of $\mathcal{C}_{g,n}^p$ consisting of simplices $\{\gamma_0, \dots, \gamma_k\}$ such that each γ_i is the homotopy class of a nonseparating simple closed curve.

For us, the key result about $\mathcal{C}_{g,n}^p$ is the following, which is due to Harer [13].

Theorem 2.7 ([13]). *Fix $g, n, p \geq 0$. The space $\mathcal{C}_{g,n}^p$ is $(2g - 2)$ -connected if $n = p = 0$ and is $(2g + n + p - 3)$ -connected if $n + p > 0$.*

We will also need the following folklore lemma (see, e.g., [21, Lemma A.2]), which while not strictly about the curve complex is in the same spirit.

Lemma 2.8. *Fix $g \geq 1$ and $n, p \geq 0$. Let γ and γ' be simple closed nonseparating curves on $\Sigma_{g,n}^p$. There then exists a sequence η_1, \dots, η_k of simple closed nonseparating curves on $\Sigma_{g,n}^p$ such that $\gamma = \eta_1$, such that $\gamma' = \eta_k$, and such that η_i and η_{i+1} intersect exactly once for $1 \leq i < k$.*

2.2.4 Some results on finite-index subgroups

Here we collect some basic results on finite-index subgroups of the mapping class group. We start with the following example, which will play a small role in our proof.

Definition. The *level L subgroup* of $\text{Mod}_{g,n}^p(L)$, denoted $\text{Mod}_{g,n}^p(L)$, is the kernel of the action of $\text{Mod}_{g,n}^p(L)$ on $H_1(\Sigma_{g,n}^p; \mathbb{Z}/L)$.

Remark. There are other possible definitions of $\text{Mod}_{g,n}^p(L)$ when $n + p \geq 2$, but the above suffices for our purposes.

Next, we will need the following theorem, which was proven independently by Bridson [7] and the second author [22].

Theorem 2.9 ([7, 22]). *Fix $g \geq 3$ and $n, p \geq 0$. Let Γ be a finite-index subgroup of $\text{Mod}_{g,n}^p$, let γ be a simple closed curve on $\Sigma_{g,n}^p$, and let $k \geq 1$ be such that $T_\gamma^k \in \Gamma$. Then the image of T_γ^k in $H_1(\Gamma; \mathbb{Q})$ is zero.*

In fact, we will need the following small extension of Theorem 2.9. A multitwist $M \in \text{Mod}_{g,n}^p$ is a product $T_{\gamma_1}^{k_1} \cdots T_{\gamma_m}^{k_m}$, where the γ_i are disjoint simple closed curves and $k_i \in \mathbb{Z}$.

Corollary 2.10. *Fix $g \geq 3$ and $n, p \geq 0$. Let Γ be a finite-index subgroup of $\text{Mod}_{g,n}^p$ and let $M \in \text{Mod}_{g,n}^p$ be a multitwist such that $M \in \Gamma$. Then the image of M in $H_1(\Gamma; \mathbb{Q})$ is zero.*

Proof. If $M = T_{\gamma_1}^{k_1} \cdots T_{\gamma_m}^{k_m}$ is a multitwist, then for some $K \geq 1$ we have $T_{\gamma_i}^{Kk_i} \in \Gamma$ for $1 \leq i \leq m$. Since the T_{γ_i} commute, we have $M^K = T_{\gamma_1}^{Kk_1} \cdots T_{\gamma_m}^{Kk_m}$. Theorem 2.9 says that the image of $T_{\gamma_i}^{Kk_i}$ in $H_1(\Gamma; \mathbb{Q})$ vanishes for $1 \leq i \leq m$, so the image of M^K (and thus M) does as well. \square

The final theorem we need is the following deep result of Boggi (cf. [3, Lemma 2.6]).

Theorem 2.11 ([5, Lemma 5.5]). *Let Γ be a finite-index subgroup of $\text{Mod}_{g,n}^p$. Then $H_k(\mathcal{C}_{g,n}^p/\Gamma; \mathbb{Q}) = 0$ for $0 \leq k \leq 2g - 2$ if $n + p = 0$ and for $0 \leq k \leq 2g + n + m - 3$ if $n + p > 0$.*

Remark. The proof of the main theorem of [5] has a fatal flaw, but the proof of Theorem 2.11 is correct. See [1] for details.

3 The mapping class group (conditionally) does not virtually surject onto \mathbb{Z}

In this section, we prove that Conjecture 1.2 implies Conjecture 1.1. The actual proof is contained in §3.3 (see Theorem 3.3). This is preceded by §3.1 and §3.2, which prove two necessary lemmas. The proof has some annoying features when $g = 3$. To avoid having to deal with this case separately, we make the following definition.

Definition. Fix $g, n, p \geq 0$, and let Γ be a subgroup of $\text{Mod}_{g,n}^p$. Let $T(\Gamma)$ be the subgroup of Γ generated by the set

$$\{M \mid M \in \text{Mod}_{g,n}^p \text{ is a multitwist, } M \in \Gamma\}$$

and define $\hat{H}_1(\Gamma; \mathbb{Q}) = H_1(\Gamma/T(\Gamma); \mathbb{Q})$.

Remark. If $g \geq 3$ and Γ is finite-index, then Corollary 2.10 implies that $\hat{H}_1(\Gamma; \mathbb{Q}) = H_1(\Gamma; \mathbb{Q})$.

3.1 Filling in punctures

In this section, we prove the following lemma, which says that the portion of a finite-index subgroup $\Gamma < \text{Mod}_{g,n}^p$ living in the point-pushing subgroup (i.e. the kernel of the Birman exact sequence) goes to zero in $\hat{H}_1(\Gamma; \mathbb{Q})$.

Lemma 3.1. *Fix $g \geq 2$ and $n \geq 0$ and $p \geq 1$. Assume that Conjecture 1.2 holds for $\Sigma_{g,n}^{p-1}$. Let $\Gamma < \text{Mod}_{g,n}^p$ be a finite-index subgroup. Fix a puncture of $\Sigma_{g,n}^p$ and thus via the Birman exact sequence a point-pushing subgroup $\pi_1(\Sigma_{g,n}^{p-1}) < \text{Mod}_{g,n}^p$. Then the map $H_1(\Gamma \cap \pi_1(\Sigma_{g,n}^{p-1}); \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma; \mathbb{Q})$ is the zero map.*

Proof. Let K be a finite-index characteristic subgroup of $\pi_1(\Sigma_{g,n}^{p-1})$ such that $K < \Gamma \cap \pi_1(\Sigma_{g,n}^{p-1})$. For example, K could be the intersection of all index $[\pi_1(\Sigma_{g,n}^{p-1}) : \Gamma \cap \pi_1(\Sigma_{g,n}^{p-1})]$ subgroups of $\pi_1(\Sigma_{g,n}^{p-1})$. Lemma 2.2 says that the map $H_1(K; \mathbb{Q}) \rightarrow H_1(\Gamma \cap \pi_1(\Sigma_{g,n}^{p-1}); \mathbb{Q})$ is surjective, so it is enough to show that the map $H_1(K; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma; \mathbb{Q})$ is the zero map. This will have two steps.

Step 1. *The map $H_1(K; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma; \mathbb{Q})$ factors through V_K .*

Let $\rho : S \rightarrow \Sigma_{g,n}^{p-1}$ be the finite cover of $\Sigma_{g,n}^{p-1}$ corresponding to K and let $B < H_1(K; \mathbb{Q}) \cong H_1(S; \mathbb{Q})$ be the boundary subspace. Thus by definition $V_K = H_1(K; \mathbb{Q})/B$, so we must show that B goes to 0 in $\hat{H}_1(\Gamma; \mathbb{Q})$. To do this, we must be careful with basepoints. Let v be the basepoint in $\Sigma_{g,n}^{p-1}$ and let $v_S \in S$ be the lift of v to S such that the image of the map $\rho_* : \pi_1(S, v_S) \rightarrow \pi_1(\Sigma_{g,n}^{p-1}, v)$ is K .

Let δ_S be either a boundary component of S or a simple closed curve in S that is freely homotopic to a puncture of S . The subspace B is generated by the homology classes of such δ_S . Chasing through the definitions, it is enough to find some $\eta_S \in \pi_1(S, v_S)$ with the following two properties.

- η_S is freely homotopic to δ_S .
- Let

$$\eta = \rho_*(\eta_S) \in K < \pi_1(\Sigma_{g,n}^{p-1}, v).$$

We then want the element of $\Gamma < \text{Mod}_{g,n}^p$ corresponding to η to be a multitwist.

By Lemma 2.5, this second property will follow if we can show that $\eta = \mu^k$ for some $\mu \in \pi_1(\Sigma_{g,n}^{p-1}, v)$ that can be realized by a based simple closed curve.

Pick a point $q_S \in \delta_S$. Let γ_1^S be a path in S from v_S to q_S and let γ_2^S be an embedded q_S -based loop in S that goes once around δ_S in the positive direction. Define $\eta_S \in \pi_1(S, v_S)$ to be the homotopy class of the path $\gamma_1^S \cdot \gamma_2^S \cdot \bar{\gamma}_1^S$. Clearly η_S is freely homotopic to δ .

Let $\eta = \rho_*(\eta_S)$ and $\delta = \rho(\delta_S)$. If δ_S is a boundary component of S , then δ is a boundary component of $\Sigma_{g,n}^{p-1}$. If δ_S is freely homotopy to a puncture of S , then after possibly modifying δ_S by a homotopy, we can assume that δ is a simple closed curve freely homotopic to a puncture of $\Sigma_{g,n}^{p-1}$. The map $\rho|_{\delta_S} : \delta_S \rightarrow \delta$ is a k -fold cover for some $k \geq 1$. Let $q = \rho(q_S) \in \delta$ and $\gamma_1 = \rho_*(\gamma_1^S)$, so γ_1 is a path in $\Sigma_{g,n}^{p-1}$ from v to q . Also, let γ_2 be an

embedded q -based loop in $\Sigma_{g,n}^{p-1}$ that goes once around δ in the positive direction. By the above, we have that

$$\eta = \rho_*(\gamma_1^S) \cdot \rho_*(\gamma_2^S) \cdot \rho_*(\overline{\gamma}_1^S) = \gamma_1 \cdot \gamma_2^k \cdot \overline{\gamma}_1 = (\gamma_1 \cdot \gamma_2 \cdot \overline{\gamma}_1)^k.$$

Setting

$$\mu = \gamma_1 \cdot \gamma_2 \cdot \overline{\gamma}_1 \in \pi_1(\Sigma_{g,n}^{p-1}, v),$$

the curve μ is freely homotopic to the simple closed curve γ_2 , so by Lemma 2.6 we get that μ can in fact be realized by a based simple closed curve, as desired.

Step 2. *The induced map $V_K \rightarrow H_1(\Gamma; \mathbb{Q})$ is the zero map.*

Since the conjugation action of Γ on itself induces the trivial action on $H_1(\Gamma; \mathbb{Q})$, the map in question factors through $(V_K)_\Gamma$. We will prove that $(V_K)_\Gamma = 0$. Since we are assuming that Conjecture 1.2 holds for $\Sigma_{g,n}^{p-1}$, we know that Γ does not fix any nonzero vector in V_K , i.e. that $(V_K)^\Gamma = 0$. As in Step 1, let S be the cover of $\Sigma_{g,n}^{p-1}$ corresponding to K . Observe that $V_K \cong H_1(S'; \mathbb{Q})$, where S' is the closed surface obtained from S by filling in all of its punctures and gluing discs to all of its boundary components. In particular, V_K is naturally isomorphic to its dual via the algebraic intersection pairing, so by Lemma 2.1 we can conclude that $(V_K)_\Gamma = 0$, as desired. \square

3.2 Everything is contained in nonseparating curve stabilizers

In this section, we prove Lemma 3.2 below, which says that if $\Gamma < \text{Mod}_{g,n}^p$ is a finite-index subgroup, then $H_1(\Gamma; \mathbb{Q})$ is “concentrated” in the stabilizers of nonseparating simple closed curves. We need the following definition.

Definition. Let Γ be a subgroup of $\text{Mod}_{g,n}^p$.

- If γ is a simple closed curve on $\Sigma_{g,n}^p$, then let

$$\Gamma_\gamma = \{g \in \Gamma \mid g \text{ fixes the isotopy class of } \gamma\}.$$

- If S is a subsurface of $\Sigma_{g,n}^p$, then let Γ_S be the intersection of Γ with the image of the natural map $\text{Mod}(S) \rightarrow \text{Mod}_{g,n}^p$.

We can now state our lemma.

Lemma 3.2. *Fix $g \geq 3$ and $n, p \geq 0$. Assume that Conjecture 1.2 holds for $\Sigma_{g-1, n+1}^p$. Let $\Gamma < \text{Mod}_{g,n}^p$ be a finite-index subgroup and let γ be a nonseparating simple closed curve on $\Sigma_{g,n}^p$. Then the natural map $H_1(\Gamma_\gamma; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q})$ is surjective.*

Proof. Lemma 2.2 implies that the vertical maps in the commutative diagram

$$\begin{array}{ccc} H_1((\Gamma \cap \text{Mod}_{g,n}^p(3))_\gamma; \mathbb{Q}) & \longrightarrow & H_1(\Gamma \cap \text{Mod}_{g,n}^p(3); \mathbb{Q}) \\ \downarrow & & \downarrow \\ H_1(\Gamma_\gamma; \mathbb{Q}) & \longrightarrow & H_1(\Gamma; \mathbb{Q}), \end{array}$$

are surjective. It is therefore enough to show that the map

$$H_1((\Gamma \cap \text{Mod}_{g,n}^p(3))_\gamma; \mathbb{Q}) \longrightarrow H_1(\Gamma \cap \text{Mod}_{g,n}^p(3); \mathbb{Q})$$

is surjective. Replacing Γ with $\Gamma \cap \text{Mod}_{g,n}^p(3)$, we can thus assume without loss of generality that $\Gamma < \text{Mod}_{g,n}^p(3)$. The proof will have four steps. For the first two, recall that $\mathcal{C}_{g,n}^p$ is the curve complex and $\mathcal{CN}\mathcal{S}_{g,n}^p$ is the nonseparating curve complex.

Step 1. *The map*

$$\bigoplus_{\gamma \in (\mathcal{C}_{g,n}^p)^{(0)}} H_1(\Gamma_\gamma; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q})$$

is a surjection.

Since $\Gamma < \text{Mod}_{g,n}^p(3)$, the group Γ acts on $\mathcal{C}_{g,n}^p$ without rotations. Since $g \geq 3$, Theorems 2.7 and 2.11 say that

$$H_1(\mathcal{C}_{g,n}^p; \mathbb{Q}) = H_1(\mathcal{C}_{g,n}^p/\Gamma; \mathbb{Q}) = 0.$$

The desired conclusion thus follows from Theorem 2.4.

Step 2. *The map*

$$\bigoplus_{\gamma \in (\mathcal{CN}\mathcal{S}_{g,n}^p)^{(0)}} H_1(\Gamma_\gamma; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q}) \tag{3}$$

is a surjection.

The proof will be by induction on $n + p$. We will treat the base case and the inductive case simultaneously, so assume that the claim is true for all smaller values of n and p .

By Step 1, we must show that for every nontrivial separating curve δ , the image of the map $H_1(\Gamma_\delta; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q})$ is contained in the image of the map (3). This is equivalent to showing that for every $\phi \in \Gamma_\delta$, there exists some $k \geq 1$ such that ϕ^k can be written as a product of mapping classes in Γ that fix nonseparating curves.

Assume that δ separates $\Sigma_{g,n}^p$ into subsurfaces T_1 and T_2 . Since $\phi \in \Gamma_\delta$ and $\Gamma < \text{Mod}_{g,n}^p(3)$, the mapping class ϕ cannot exchange T_1 and T_2 (up to homotopy). In other words, we can write $\phi = f_1 \cdot f_2$, where $f_i \in (\text{Mod}_{g,n}^p)_{T_i}$ (in other words, f_i is “supported” on T_i). Since Γ is a finite-index subgroup of $\text{Mod}_{g,n}^p$, there exists some $k \geq 1$ such that $f_i^k \in \Gamma$ for $i = 1, 2$. The mapping classes f_1 and f_2 commute, so $\phi^k = f_1^k \cdot f_2^k$.

There are now two cases. If both T_1 and T_2 have positive genus, then letting γ_i be a nonseparating simple closed curve in T_i , we have $f_1^k \in \Gamma_{\gamma_2}$ and $f_2^k \in \Gamma_{\gamma_1}$, as desired. If one of the T_i (without loss of generality, T_1) has genus 0, then the genus of T_2 is at least 3. Also, since δ is nontrivial, T_2 has fewer than $n + p$ punctures/boundary components. Thus the inductive hypothesis says that after possibly increasing k , we have that f_2^k can be written as a product of mapping classes in Γ that fix nonseparating curves. Also, letting γ_2 be a nonseparating curve in T_2 , we have that f_1^k fixes γ_2 , and we are done.

Step 3. *Let S be an embedded subsurface of $\Sigma_{g,n}^p$ such that $S \cong \Sigma_{g-1,n+1}^p$. Let γ be a nonseparating curve on $\Sigma_{g,n}^p$ that lies entirely within $\Sigma_{g,n}^p \setminus \text{Int}(S) \cong \Sigma_{1,1}$ (see Figure 2.a). Then the natural map*

$$\hat{H}_1(\Gamma_S; \mathbb{Q}) \longrightarrow \hat{H}_1(\Gamma_\gamma; \mathbb{Q})$$

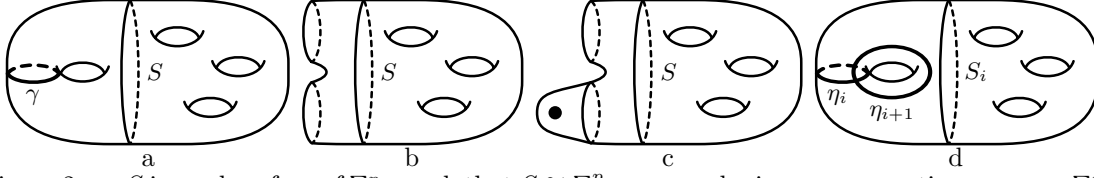


Figure 2: a. S is a subsurface of $\Sigma_{g,n}^p$ such that $S \cong \Sigma_{g-1,n+1}^p$ and γ is a nonseparating curve on $\Sigma_{g,n}^p$ that lies entirely within $\Sigma_{g,n}^p \setminus \text{Int}(S) \cong \Sigma_{1,1}$. b. The complement X of a regular neighborhood of γ . c. The result X' of gluing a punctured disc to X . d. S_i is the complement of a regular neighborhood of $\eta_i \cup \eta_{i+1}$.

is a surjection.

We make the following definitions.

- Let X be the complement of a regular neighborhood of $\gamma \subset \Sigma_{g,n}^p$ (see Figure 2.b), so $X \cong \Sigma_{g-1,n+2}^p$.
- Let X' be the result of gluing a punctured disc to one of the two boundary components of X that are not boundary components of $\Sigma_{g,n}^p$ (see Figure 2.c), so $X' \cong \Sigma_{g-1,n+1}^{p+1}$.
- Let X'' be the result of filling in the puncture on the glued-on disc in X' , so $X'' \cong \Sigma_{g-1,n+1}^p$.

We have an embedding $S \hookrightarrow \Sigma_{g,n}^p$. As shown in Figures 2.a–c, we can arrange for there to exist embeddings $S \hookrightarrow X$ and $S \hookrightarrow X'$ and $S \hookrightarrow X''$ such that the diagram

$$\begin{array}{ccccc}
 & \Sigma_{g,n}^p & \longleftarrow & X & \longrightarrow & X' & \longrightarrow & X'' \\
 & \uparrow & & \nearrow & \nearrow & \nearrow & & \nearrow \\
 S & & & & & & &
 \end{array}$$

commutes. There is an induced commutative diagram

$$\begin{array}{ccccccc}
 (\text{Mod}_{g,n}^p)_\gamma & \longleftarrow & \text{Mod}(X) & \longrightarrow & \text{Mod}(X') & \longrightarrow & \text{Mod}(X'') \\
 & \uparrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow \\
 \text{Mod}(S) & & & & & &
 \end{array}$$

of mapping class groups. Define $\Gamma_X \subset \text{Mod}(X)$ to be the pullback of $\Gamma_\gamma \subset (\text{Mod}_{g,n}^p)_\gamma$ to $\text{Mod}(X)$. Also, define Γ'_X to be the image of Γ_X in $\text{Mod}(X')$ and Γ''_X to be the image of Γ'_X in $\text{Mod}(X'')$. We then have a commutative diagram

$$\begin{array}{ccccccc}
 \Gamma_\gamma & \longleftarrow & \Gamma_X & \longrightarrow & \Gamma'_X & \longrightarrow & \Gamma''_X \\
 & \uparrow & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow \\
 \Gamma_S & & & & & &
 \end{array}$$

Passing to first homology, we have a commutative diagram

$$\begin{array}{ccccccc}
\hat{H}_1(\Gamma_\gamma; \mathbb{Q}) & \longleftarrow & \hat{H}_1(\Gamma_X; \mathbb{Q}) & \longrightarrow & \hat{H}_1(\Gamma'_X; \mathbb{Q}) & \longrightarrow & \hat{H}_1(\Gamma''_X; \mathbb{Q}) \\
\uparrow & & \nearrow & \nearrow & \nearrow & & \\
H_1(\Gamma_S; \mathbb{Q}) & & & & & &
\end{array}$$

Our goal is to show that the map $H_1(\Gamma_S; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma_\gamma; \mathbb{Q})$ is a surjection. To do this, it is enough to show that all the maps on the first row of this commutative diagram are isomorphisms and that the map $H_1(\Gamma_S; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma''_X; \mathbb{Q})$ is a surjection. We deal with each of these claims in turn.

Claim 1. The map $\hat{H}_1(\Gamma_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma_\gamma; \mathbb{Q})$ is an isomorphism.

Restricting exact sequence (2) to Γ_X , we obtain a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_X \longrightarrow \Gamma_\gamma \longrightarrow 1.$$

Here \mathbb{Z} is generated by $(T_x T_y^{-1})^k = T_x^k T_y^{-k}$, where x and y are the boundary components of $\Sigma_{g,n}^b \setminus \text{Int}(X)$ and $k \geq 1$. Since the kernel \mathbb{Z} is generated by a multitwist, we conclude that the map $\hat{H}_1(\Gamma_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma_\gamma; \mathbb{Q})$ is an isomorphism.

Claim 2. The map $\hat{H}_1(\Gamma_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma'_X; \mathbb{Q})$ is an isomorphism.

Restricting exact sequence (1) to Γ_X , we obtain a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \Gamma_X \longrightarrow \Gamma'_X \longrightarrow 1.$$

Here \mathbb{Z} is generated by T_β^l , where β is the boundary component of X to which we are gluing a punctured disc to obtain X' and $l \geq 1$. Since the kernel \mathbb{Z} is generated by a multitwist, we conclude that the map $\hat{H}_1(\Gamma_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma'_X; \mathbb{Q})$ is an isomorphism.

Claim 3. The map $\hat{H}_1(\Gamma'_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma''_X; \mathbb{Q})$ is an isomorphism.

We have a Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g-1,n+1}^p) \longrightarrow \text{Mod}(X') \longrightarrow \text{Mod}(X'') \longrightarrow 1. \quad (4)$$

Restricting this to Γ'_X , we get a short exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g-1,n+1}^p) \cap \Gamma'_X \longrightarrow \Gamma'_X \longrightarrow \Gamma''_X \longrightarrow 1. \quad (5)$$

Lemma 3.1 says that the map $H_1(\pi_1(\Sigma_{g-1,n+1}^p) \cap \Gamma'_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma'_X; \mathbb{Q})$ is the zero map, so we conclude that the map $\hat{H}_1(\Gamma'_X; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma''_X; \mathbb{Q})$ is an isomorphism.

Claim 4. The map $H_1(\Gamma_S; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma''_X; \mathbb{Q})$ is a surjection.

Since the map $\text{Mod}(S) \rightarrow \text{Mod}(X'')$ is an isomorphism, one might first think that the map $\Gamma_S \rightarrow \Gamma_X''$ is an isomorphism. However, this need not hold. Indeed, composing the isomorphism $\text{Mod}(X'') \rightarrow \text{Mod}(S)$ with the map $\text{Mod}(S) \rightarrow \text{Mod}(X')$ gives a splitting of exact sequence (4). If the map $\Gamma_S \rightarrow \Gamma_X''$ were an isomorphism, then in a similar way we would obtain a splitting of exact sequence (5), which need not split. However, chasing through the definitions, we see that the image of Γ_S in Γ_X'' is a finite-index subgroup, so Lemma 2.2 implies that the map $H_1(\Gamma_S; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma_X''; \mathbb{Q})$ is a surjection.

Step 4. Let γ be a nonseparating curve on $\Sigma_{g,n}^p$. Then the natural map $H_1(\Gamma_\gamma; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q})$ is a surjection.

Let γ' be another nonseparating curve on $\Sigma_{g,n}^p$. By Step 2, it is enough to show that the images of the maps

$$H_1(\Gamma_\gamma; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q}) \quad \text{and} \quad H_1(\Gamma_{\gamma'}; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q}) \quad (6)$$

are the same. By Lemma 2.8, there is a sequence η_1, \dots, η_k of simple closed nonseparating curves on $\Sigma_{g,n}^p$ such that $\gamma = \eta_1$, such that $\gamma' = \eta_k$, and such that η_i and η_{i+1} intersect exactly once for $1 \leq i < k$. To show that the images of the maps in (6) are the same, it is thus enough to show that the images of the maps

$$H_1(\Gamma_{\eta_i}; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q}) \quad \text{and} \quad H_1(\Gamma_{\eta_{i+1}}; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q}) \quad (7)$$

are the same for $1 \leq i < k$. Since $g \geq 3$, Corollary 2.10 implies that these two maps factor through $\hat{H}_1(\Gamma_{\eta_i}; \mathbb{Q})$ and $\hat{H}_1(\Gamma_{\eta_{i+1}}; \mathbb{Q})$, respectively. Let S_i be the complement of a regular neighborhood of $\eta_i \cup \eta_{i+1}$ (see Figure 2.d). We thus have $S \cong \Sigma_{g-1,n+1}^p$ and $\eta_i, \eta_{i+1} \subset \Sigma_{g,n}^p \setminus \text{Int}(S_i) \cong \Sigma_{1,1}$. We have a commutative diagram

$$\begin{array}{ccc} \Gamma_{S_i} & \longrightarrow & \Gamma_{\eta_i} \\ \downarrow & & \downarrow \\ \Gamma_{\eta_{i+1}} & \longrightarrow & \Gamma \end{array}$$

Step 3 says that the maps $\hat{H}_1(\Gamma_{S_i}; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma_{\eta_i}; \mathbb{Q})$ and $\hat{H}_1(\Gamma_{S_i}; \mathbb{Q}) \rightarrow \hat{H}_1(\Gamma_{\eta_{i+1}}; \mathbb{Q})$ are surjections, so we conclude that the maps in (7) have the same image, as desired. \square

3.3 The proof

We finally are in a position to prove that Conjecture 1.2 implies Conjecture 1.1.

Theorem 3.3. Fix $g \geq 3$ and $n, p \geq 0$. Assume that Conjecture 1.2 holds for $\Sigma_{g-1,n+1}^p$. Let $\Gamma < \text{Mod}_{g,n}^p$ be a finite-index subgroup. Then $H_1(\Gamma; \mathbb{Q}) = 0$.

Proof. By Lemma 2.2, we can assume without loss of generality that Γ is a normal subgroup of $\text{Mod}_{g,n}^p$. Lemma 2.3 then implies that

$$H_1(\text{Mod}_{g,n}^p; \mathbb{Q}) \cong (H_1(\Gamma; \mathbb{Q}))_{\text{Mod}_{g,n}^p}.$$

Since $g \geq 3$, the group $\text{Mod}_{g,n}^p$ is perfect, so $H_1(\text{Mod}_{g,n}^p; \mathbb{Q}) = 0$. It is thus enough to show that $\text{Mod}_{g,n}^p$ acts trivially on $H_1(\Gamma; \mathbb{Q})$. The group $\text{Mod}_{g,n}^p$ is generated by Dehn twists T_γ about nonseparating curves γ . It is thus enough to show that T_γ acts trivially on $H_1(\Gamma; \mathbb{Q})$ for γ nonseparating. Clearly the conjugation action of T_γ on Γ restricts to the identity on the subgroup $\Gamma_\gamma < \Gamma$, so the desired result is an immediate consequence of Lemma 3.2, which says that the map $H_1(\Gamma_\gamma; \mathbb{Q}) \rightarrow H_1(\Gamma; \mathbb{Q})$ is surjective for all nonseparating curves γ . \square

4 A (conditional) construction of finite-index subgroups of the mapping class group that surject onto \mathbb{Z}

We close the paper by proving if Conjecture 1.2 is false, then Conjecture 1.1 is also false. Assume, therefore, that $g \geq 2$ and $n \geq 1$ and $p \geq 0$. Let $K < \pi_1(\Sigma_{g,n}^p)$ be a counterexample to Conjecture 1.2. There thus exists a nonzero vector $v_0 \in V_K$ such that the $\text{Mod}_{g,n}^{p+1}$ -orbit of v_0 is finite. Our goal is to find a finite-index subgroup $\Gamma < \text{Mod}_{g,n}^{p+1}$ such that $H_1(\Gamma; \mathbb{Q}) \neq 0$.

Consider the Birman exact sequence

$$1 \longrightarrow \pi_1(\Sigma_{g,n}^p) \longrightarrow \text{Mod}_{g,n}^{p+1} \longrightarrow \text{Mod}_{g,n}^p \longrightarrow 1.$$

Since $n \geq 1$, this exact sequence splits. Fixing a splitting, we obtain an isomorphism

$$\text{Mod}_{g,n}^{p+1} \cong \pi_1(\Sigma_{g,n}^p) \rtimes \text{Mod}_{g,n}^p. \quad (8)$$

We remark that the action of $\text{Mod}_{g,n}^p$ on $\pi_1(\Sigma_{g,n}^p)$ in (8) is *not* natural (it depends on the choice of splitting).

Let $\rho : \text{Mod}_{g,n}^{p+1} \rightarrow \text{Aut}(V_K)$ be the higher Prym representation. Since the orbit of $v_0 \in V_K$ is finite, we can find a finite-index subgroup $G' < \text{Mod}_{g,n}^{p+1}$ such that $\rho(g)(v_0) = v_0$ for all $g \in G'$. Regarding $\text{Mod}_{g,n}^p$ as a subgroup of $\text{Mod}_{g,n}^{p+1}$ via the decomposition (8), let $G = G' \cap \text{Mod}_{g,n}^p$. Since $K < \pi_1(\Sigma_{g,n}^p)$ is characteristic, we can form the finite-index subgroup

$$\Gamma = K \rtimes G < \pi_1(\Sigma_{g,n}^p) \rtimes \text{Mod}_{g,n}^p \cong \text{Mod}_{g,n}^{p+1}.$$

From its semidirect product decomposition, we get that

$$H_1(\Gamma; \mathbb{Q}) \cong H_1(G; \mathbb{Q}) \oplus (H_1(K; \mathbb{Q}))_G.$$

To prove that $H_1(\Gamma; \mathbb{Q}) \neq 0$, it is thus enough to prove that $(H_1(K; \mathbb{Q}))_G \neq 0$. Dually, by Lemma 2.1 it is enough to construct a nonzero homomorphism $\psi : H_1(K; \mathbb{Q}) \rightarrow \mathbb{Q}$ which is invariant under the natural G -action on $\text{Hom}(H_1(K; \mathbb{Q}), \mathbb{Q})$.

Recall that $V_K = H_1(K; \mathbb{Q})/B$ where B is the boundary subspace of $H_1(K; \mathbb{Q})$. It follows that V_K is the first rational homology group of the closed surface that results from taking the finite cover of $\Sigma_{g,b}^p$ corresponding to $K < \pi_1(\Sigma_{g,b}^p)$ and gluing discs to all of its boundary components and filling in all of its punctures. In particular, V_K is a symplectic

vector space, i.e. it has a nondegenerate alternating pairing $i : V_K \times V_K \rightarrow \mathbb{Q}$, namely the algebraic intersection form. Define $\psi : H_1(K; \mathbb{Q}) \rightarrow \mathbb{Q}$ to be the composition

$$H_1(K; \mathbb{Q}) \longrightarrow V_K \xrightarrow{i(v_0, \cdot)} \mathbb{Q}.$$

Since i is nondegenerate and $v_0 \neq 0$, the map ψ is nonzero. Also, since v_0 is invariant under G , it follows that ψ is invariant under G , as desired.

References

- [1] D. Abramovich, review of [5], Math Reviews, MR2242960 (2008f:14037).
- [2] J. Anderson, Mapping class groups do not have Kazhdan’s property (T), preprint 2007.
- [3] E. Arbarello and M. Cornalba, Calculating cohomology groups of moduli spaces of curves via algebraic geometry, Publ. Math. Inst. Hautes Études Sci. **88** (1998), 97–127.
- [4] J. S. Birman, Mapping class groups and their relationship to braid groups, Comm. Pure Appl. Math. **22** (1969), 213–238.
- [5] M. Boggi, Profinite Teichmüller theory, Math. Nachr. **279** (2006), no. 9-10, 953–987.
- [6] M. Boggi, Fundamental groups of moduli stacks of stable curves of compact type, Geom. Topol. **13** (2009), 247–276.
- [7] M. Bridson, Semisimple actions of mapping class groups on CAT(0) spaces, to appear.
- [8] K. S. Brown, *Cohomology of groups*, Corrected reprint of the 1982 original, Springer, New York, 1994.
- [9] P. Deligne, Poids dans la cohomologie des variétés algébriques, Proc. International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 79–85, Canad. Math. Congress, Montreal, Que., 1975.
- [10] B. Farb and D. Margalit, A primer on mapping class groups, preprint 2010.
- [11] F. Grunewald and A. Lubotzky, Linear representations of the automorphism group of a free group, Geom. Funct. Anal. **18** (2009), no. 5, 1564–1608.
- [12] R. M. Hain, Torelli groups and geometry of moduli spaces of curves, in *Current topics in complex algebraic geometry (Berkeley, CA, 1992/93)*, 97–143, Cambridge Univ. Press, Cambridge.
- [13] J. L. Harer, Stability of the homology of the mapping class groups of orientable surfaces, Ann. of Math. (2) **121** (1985), no. 2, 215–249.
- [14] W. J. Harvey, Geometric structure of surface mapping class groups, in *Homological group theory (Proc. Sympos., Durham, 1977)*, 255–269, London Math. Soc. Lecture Note Ser., 36 Cambridge Univ. Press, Cambridge.
- [15] N. V. Ivanov, Fifteen problems about the mapping class groups, in *Problems on mapping class groups and related topics*, 71–80, Proc. Sympos. Pure Math., 74, Amer. Math. Soc., Providence, RI.
- [16] D. Johnson, The structure of the Torelli group. I. A finite set of generators for \mathcal{I} , Ann. of Math. (2) **118** (1983), no. 3, 423–442.

- [17] R. Kirby, Problems in low dimensional manifold theory, in *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, 273–312, Proc. Sympos. Pure Math., XXXII Amer. Math. Soc., Providence, RI.
- [18] E. Looijenga, Prym representations of mapping class groups, *Geom. Dedicata* **64** (1997), no. 1, 69–83.
- [19] J. D. McCarthy, On the first cohomology group of cofinite subgroups in surface mapping class groups, *Topology* **40** (2001), no. 2, 401–418.
- [20] C. McMullen, Braid groups and Hodge theory, *Math. Ann.*, to appear.
- [21] A. Putman, Cutting and pasting in the Torelli group, *Geom. Topol.* **11** (2007), 829–865.
- [22] A. Putman, A note on the abelianizations of finite-index subgroups of the mapping class group, *Proc. Amer. Math. Soc.* **138** (2010), no. 2, 753–758.
- [23] F. Taherkhani, The Kazhdan property of the mapping class group of closed surfaces and the first cohomology group of its cofinite subgroups, *Experiment. Math.* **9** (2000), no. 2, 261–274.
- [24] R. J. Zimmer, *Ergodic theory and semisimple groups*, Birkhäuser, Basel, 1984.

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